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Characterizing complexity of non-invertible chaotic maps in the Shannon-Fisher information plane with ordinal patterns

David Spichak^a, Audrey Kupetsky^b, Andrés Aragoneses^{a,*}

^a Department of Physics, Eastern Washington University, Cheney, WA 99004, USA ^b Physics and Astronomy department, Carleton College, Northfield, MN 55057, USA

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ABSTRACT

Being able to distinguish the different types of dynamics present in a given nonlinear system is of great importance in complex dynamics. It allows to characterize the system, find similarities and differences with other nonlinear systems, and classify those dynamical regimes to understand them better. For systems that develop chaos it is not always easy to distinguish determinism from stochasticity. We analyze several non-invertible maps by projecting them on the two-dimensional Fisher-Shannon plane using ordinal patterns. We find that this technique unfolds the complex structure of chaotic systems, showing more details than other methods. It also reveals signatures common to most of the non-invertible maps, and demonstrates its capability to distinguish determinism from stochasticity.

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1. Introduction

Finding a quantifier to characterize complexity is a task that has been under discussion for several decades. Different researchers have found different quantifiers that are informative for some types of systems but are not helpful for others [1–5]. One of the most successful ways to quantify complexity is represented by the causal Shannon-Fisher plane [6], where the dynamics of the complex system is mapped into a 2D plane, represented through its Shannon entropy and its Fisher's Information Measure [7–10]. This representation is convenient when comparison among different complex dynamical systems is required. It represents, in a simple visual manner, local and global information associated with the dynamical system.

Shannon entropy, S, based on the PDF (probability distribution function) of a time series, reflects the global behavior of the dynamics, and is robust to changes in the distribution on small scales. It quantifies how far or close the system is with respect to an uncorrelated stochastic process; how ordered or random its dynamics is. It can be used to quantify how much one can predict the next value of the time series. Normalized Shannon entropy is defined as

Corresponding author. E-mail address: aaragoneses@ewu.edu (A. Aragoneses).

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$$S = -\frac{1}{\ln(N)} \sum_{i=1}^{N} p_i \ln p_i \tag{1}$$

where $\{p_i : i = 1, 2, ..., N\}$ is the probability distribution, and $0 \le N$ $S \leq 1$. N is the number of compartments the PDF has been divided into. The PDF can be generated from the values of the time series, but additional information can be gathered if the raw time series is transformed into an ordinal patterns time series [11-13], or into a network [9,14,15], previous to calculating its Shannon entropy.

In 2002 Bandt and Pompe [16] introduced a novel method to compute the entropy of a time series through a symbolic analysis, permutation entropy (PE). In order to find temporal correlations in the dynamics, they considered consecutive values of the time series and, comparing their relative magnitudes, transformed them into a sequence of ordinal patterns. From the probabilities of each of the ordinal patterns one can compute the permutation entropy. This method of calculating the entropy is very robust to experimental noise, is stable under local changes, unveils long temporal correlations in the dynamics, and allows to forecast events in a time series [5,17-20].

In order to compute the ordinal patterns from the time series they are compared the magnitudes of D consecutive events in the time sequence $\{x_i\}$. For ordinal patterns of dimension D = 2only two patterns can be formed, i.e., '01' if $x_i < x_{i+1}$, and '10' if $x_i > x_{i+1}$. Six patterns can be formed for dimension D = 3, i.e., '012'

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if $x_i < x_{i+1} < x_{i+2}$, '021' if $x_i < x_{i+2} < x_{i+1}$, and so on. The permutation entropy is then defined as

$$PE = -\frac{1}{\ln(D!)} \sum_{i=1}^{D!} p_i \ \ln p_i \tag{2}$$

where D! is the number of ordinal patterns of dimension D, and p_i is the probability of the *i*th ordinal pattern. *PE* is normalized so that $0 \le PE \le 1$. For a purely random process the normalized entropy is maximum, PE = 1, while PE = 0 corresponds to a completely predictable process where only one ordinal pattern is present. Low values of *PE* indicate deterministic structure in the dynamics as some patterns are more likely than others.

Fisher's Information Measure (FIM) [21] is another measure to quantify complexity. It represents a measure of the gradient of the distribution, and therefore is very sensitive to small, localized changes. It measures the rate of change of the consecutive values in a time series. It can be interpreted as the amount of information that can be extracted from a system. For a purely ordered system FIM is maximum, while for a stochastic process FIM is zero. It has shown to be a powerful tool to detect and describe complex behaviors in dynamical systems [22–26].

For a discrete times series FIM can be defined as [7,9,26–28]

$$FIM = F_0 \sum_{i=1}^{N-1} \left((p_{i+1})^{1/2} - (p_i)^{1/2} \right)^2 ;$$
(3)

where $F_0 = 1$ if $p_{i^*} = 1$ for $i^* = 1$ or $i^* = N$, and $p_i = 0 \quad \forall i \neq i^*$. $F = \frac{1}{2}$ otherwise.

N is the number of possible states of the system, and p_i (i = 1, 2, 3, ..., N) is a discrete probability distribution set. FIM complements entropy as a quantifier of complexity as it can detect some local changes in behavior, there where entropy cannot.

Shannon entropy and FIM have recently been used combined in the so-called Shannon–Fisher complexity plane [7,26,29]. This plane maps simultaneously Shannon's entropy and FIM to characterize the dynamical complexity of a system. It separates and compares different regions the chaotic dynamics of different dynamical systems. It also shows trends that help characterize the different chaotic regions as the control parameter of a dynamical system is varied, and helps detect transitions and bifurcations [8–10].

Wang and Shang [10] used information entropy in the Fisher–Shannon plane to find more structure in stock markets dynamics. Ravetti et al. [9] used the Fisher–Shannon plane, together with the horizontal visibility graph [14], to study the effect of noise is several chaotic maps, and characterize the transition from chaos to noise.

Olivares et al. [7,8] studied the Shannon–Fisher plane using Band–Pompe ordinal patterns of dimension D = 6, and showed that the way the ordinal patterns are sorted has an effect on the final calculated FIM. Because FIM measures the differences in consecutive values, when one calculates the D! possible ordinal patterns, they need to be sorted, and the protocol to sort them will change those differences. They compared the effect of two different sorting protocols, Lehmer vs. Keller (see [7] for details). They found that for some regions of the dynamics, the different protocols used showed a linear relationship between FIM_L and FIM_K , but not in other regions.

Here we compute the Shannon–Fisher complexity plane using Bandt–Pompe's ordinal patterns (OPs) analysis for various chaotic maps. We use OPs of different dimensions ($2 \le D \le 8$) to calculate the permutation entropy (*PE*), and find that there is an optimum dimension to characterize the dynamics. To compute FIM we use Lehmer sorting protocol. Lehmer protocol is an extended way of sorting the ordinal patterns in the bibliography and Olivares et al. [7] found that it deploys more structure for D = 6 for the logistic map than Keller's sorting protocol. We also compute the Shannon–

Fisher plane using the PDFs of the raw time series to calculate Shannon entropy and FIM. We compare both approaches and find that the OP approach is a better tool to classify and distinguish between the complex dynamics of chaotic systems. We project different non-invertible maps in the Shannon–Fisher plane and find that most of them cover the same locations on the plane, i.e., there is a signature path on the plane characteristic to most non-invertible maps.

We use the same non-invertible maps as those used by Ravetti et al. [9]. In their paper they used the horizontal visibility graph, combined with the Fisher Information Measure and Shannon entropy, to distinguish noise from chaos in different iterative maps. They could separate the deterministic iterative maps from noise for the specific control parameter values they used. Here we go one step beyond. We unveil and characterize the different dynamic regions in the maps, for a wide range of the control parameters.

2. The logistic map

We first analyze the logistic map, $x_{n+1} = rx_n(1 - x_n)$. We use the OPs approach to calculate PE and FIM. Fig. 1(a) shows the logistic map bifurcation diagram for $3.6 \le r \le 4.0$, using a color scale to indicate the *r* value, and signify regions of distinct behavior. Fig. 1(b) shows the OP-FIM-PE plane, calculated using ordinal patterns of dimension D = 4. This plane reflects different behaviors for different regions of the control parameter r. For the lowest values of r considered (blue dots), we can distinguish one region where the system lies on a straight line in the plane, of sharp decrease of FIM while *PE* changes slightly (0.45 < PE < 0.50). The system then, as r is increased, shows a curve that increases both FIM and PE. In this curve there are some legs, some escapes from that behavior that increase FIM and decrease PE. These legs correspond to the windows of periodicity in the bifurcation diagram (around r = 3.63and r = 3.74, ...). The system then enters into the large period doubling route to chaos around r = 3.83 (green dots) indicated by a curve towards higher FIM and lower PE, and then back to lower FIM and higher PE.

As *r* is increased further into the chaotic region (orange and yellow), the system moves to the bottom-right corner of the plane, of low FIM and high *PE*. Remember that a random process would have PE = 1 and FIM = 0. In this chaotic area we can still distinguish different behaviors. The plane shows different clusters of similar FIM and *PE* values. Between the clusters the system shows some clear distinction, with some sort of bifurcation in the dynamics that can be tracked to the smaller windows of periodicity in the bifurcation diagram.

This Shannon–Fisher plane computed with OPs clearly distinguishes the different dynamical regions of the logistic map in different regions in the plane, but also displays different relations between FIM and *PE* depending on the region.

Fig. 1 also compares the Shannon–Fisher planes as computed with OPs of dimension 4, Fig. 1(b), with that computed using the PDFs of the time series for the logistic map, Fig. 1(c). For this latter, 200 bins were used for the PDF of the time series. While FIM and *PE* show some characteristic correlations on the plane when computed using OPs (as described previously, for different values of *r* they are grouped in different, well-defined, regions on the plane), no clear relation is appreciable when computed with the PDFs. The different values are distributed without showing any clear spacial pattern on the plane.

The regions found with the ordinal patterns are not visible in the PDF-Fisher–Shannon plane, no structure is distinguishable. Here, Fisher Information Measure lies between 0 and 0.15, and Shannon entropy between 0.6 and 1.0. Besides the windows of periodicity, indicated by the few scattered points in the upper left region of the PDF-plane, the logistic map restricts to a much

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Fig. 1. (a) Bifurcation diagram of the logistic map. (b) Fisher–Shannon plane using ordinal patterns, OPs. (c) Fisher–Shannon plane using the PDF of the time series. Control parameter range is $3.6 \le r \le 4.0$. We use a color scale to signify regions of distinct behavior. The OP-FIM and *PE* are computed using ordinal patterns of dimension D = 4. The leftmost part of the bifurcation diagram (blue) encompasses the points in the OP-FIM-PE plane that make up the curve feature that sharply decreases FIM while *PE* remains with little change. As the *r* parameter is increased the system in the OP-FIM-PE plane follows a curved behavior that increases FIM and *PE*. Some legs of increasing FIM and decreasing *PE* correspond to the visits to the windows of periodicity in the bifurcation diagram. The chaotic region for the highest values of *r* (orange and yellow dots) move in the low FIM high *PE* region. In (c) there is no clear structure that helps differentiate the different dynamical regimes of the system. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)



Fig. 2. OP-FIM-PE plane for the logistic map for embedding dimensions of the ordinal patterns D = 2 through D = 8. For D = 2 and D = 3 there is no clear structure and the system lies in the bottom-right part of the plane. For dimensions D > 3 the plane unveils the structure of the different regimes in the dynamics of the logistic map. As we increase the dimension of the OPs the distribution of the system on the plane presents more structure, but for D = 5, 6, 7, there is no qualitative difference. (h) shows all the dimensions on the same plane. (d) through (g) show the same limits for comparison purposes.

narrower region, 0 < FIM < 0.05, and 0.8 < entropy < 1.0. On the other hand, with the OP analysis, OP-FIM extends its range to between 0.65 and 1.0, and *PE* between 0.45 and 0.75. All this indicates that the OP-Fisher–Shannon plane for the logistic map gives more information, and distinguishes better the complex dynamics, helping characterize it.

To see the relevance of the embedded dimension *D* of the ordinal patterns in computing OP-FIM and *PE*, we have calculated the OP-Fisher–Shannon plane for dimensions D = 2 through D = 8 (see Fig. 2). The quantifiers for low dimensions (D = 2, 3) do not show the full complex structure of the dynamical system. Higher dimensions are required to unveil the complex patterns hidden in the

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Fig. 3. Bifurcation maps (a–c) and OP-FIM-PE planes (d–f) for other non-invertible maps (tent, cusp, and sine). (g) OP-Fisher–Shannon plane for the four non-invertible maps. All four systems occupy the same area in the plane, they cover the same path. OPs have been computed for D = 5. (h) OP-Fisher–Shannon plane for Richer's and cubic maps, compared to the logistic map. Logistic, cusp, sine, Ricker's population, and cubic maps have identical structure on the plane. OPs of D = 7 are computed. The parameters used for the maps are those of ref. [9].

dynamics. These higher dimensions detect structures on the plane, with a trend towards higher FIM and lower *PE* as the dimensions increase (move to the upper-left region of the plane).

For dimensions D = 5-8 the change is only quantitative, not qualitative. Similar regions are equally differentiated for corresponding r values, although at slightly different points on the plane. This implies that similar information can be gained by computing D = 5 or D = 8. Because of the computational effort involved in computing OPs of high dimensions, and the high number of patterns necessary to have significant statistics, it might not be worth going higher than dimension five or six.

3. Non-invertible maps

The structure present in the OP-Fisher–Shannon plane is not characteristic only of the logistic map. We compute the same analysis for other non-invertible maps:

- Tent map: $x_{n+1} = r \min\{x_n, 1 x_n\}$, for $1 \le r \le 2$.
- Sine map: $x_{n+1} = r \sin(\pi x_n)$, for $0 \le r \le 1$.
- Cusp map: $x_{n+1} = 1 r\sqrt{|x_n|}$, for $0 \le r \le 1$.

For these maps we find similar structure and correlations between FIM and *PE*, on the same regions of the OP-FIM-PE plane. Fig. 3 shows the bifurcation maps for the tent, cusp, and sine maps, and their corresponding OP-Fisher–Shannon planes, computed with OPs of dimension $D = 5^{\circ}$. Fig. 3(g) plots the OP-FIM *PE* for these four non-invertible maps on the same plane.

Even though the OP-FIM-*PE* plane for the maps in Fig. 3(d,e,f) seem different at first glance, Fig. 3(g) shows how they all occupy the same well-defined region on the plane. Fig. 3(g) displays the fingerprint of these non-invertible maps on the plane. All of them follow the same path with small variations, related to intrinsic characteristics of each one of them. The cusp map shows the most simple arrangement on the plane, and the other maps show some more structure, all of it on top and around that of the cusp.

The logistic and the sine maps, despite coming from very different mathematical expressions they deploy the same perioddoubling route to chaos, and similar bifurcation maps. They cover the exact same places in the OP-Fisher–Shannon plane.

The tent and cusp maps do not develop the same route to chaos that the logistic or sine maps do; there is no period-doubling, and no clear distinctions in most of their bifurcation map. Nevertheless, the OP-FIM-PE plane manages to distinguish some organization and evolution in the dynamics. In the cusp map it unveils two different regions in its dynamics. The point of maximum entropy corresponds to a kink in the plane, indicating a change in the be-

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Fig. 4. (a) PDF-Fisher–Shannon plane for the tent, sine, cusp, cubic, Ricker's population, and linear congruential generator. PDFs are computed using 200 bins. (b) Region around $PE \approx 1$, $FIM \approx 0$ enlarged.



Fig. 5. (a) Bifurcation diagram of the linear congruential generator computed with ordinal patterns of dimension D = 7. (b) Bifurcation diagram of the Pinchers map. (c,f) Time series of the Pinchers map for before the kink, S = 2.5, and after the kink, S = 3.4. (d) OP-FIM-PE plane for the linear congruential generator (D = 7) in color code. The logistic map (in red) is also shown for comparison. (e) OP-FIM-PE plane for the Pinchers map (D = 7) in color code. The logistic map (in red) is also shown for comparison. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

havior that is not recognizable by simply exploring the bifurcation map.

Other non-invertible maps (cubic map, Ricker's population model, linear congruential generator) have been studied:

- Cubic map: $x_{n+1} = r x_n (1 x_n^2)$, for $1 \le r \le 2.6$.
- Ricker's population model: $x_{n+1} = r x_n e^{-x_n}$, for $0 \le r \le 20$.
- Linear congruential generator: $x_{n+1} = A x_n + B \mod(C)$, for $0 \le A \le 4$, B = 54773, and C = 259, 200.

The bifurcation diagram of the cubic and Ricker's population maps (not shown) present similar period-doubling routes to chaos as the logistic map, and their representation on the OP-FIM-PE plane is identical to those of the logistic and the sine maps. Fig. 3(h) shows the OP-FIM-PE plane for these other non-invertible maps, where they are compared to the maps in Fig. 3(g). Here we used OPs of dimension D = 7. For lower dimensions the agreement between Ricker's and the cubic maps was not as clear as for D = 7. As we increase the dimension of the OPs all these maps tend to the same signature, the same fingerprint on the map, although some maps show that overlap for smaller dimensionality. Going to

higher dimensions means that we are exploring longer temporal correlations in the complex dynamics.

All six iterative maps represented share the same path on the OP-FIM-PE plane, while four of them (the ones with a period-doubling route to chaos) lie right on top of each other.

Just as for the logistic map, the PDF-Fisher–Shannon plane for the other non-invertible maps is much less informative than the plane based on ordinal patterns. Except for the values corresponding to the more periodic dynamics, all non-invertible maps lie in a very narrow region on the PDF plane, and do not show any differentiation as r varies (see Fig. 4). The FIM and Shannon entropy have been computed by dividing the time series into 200 bins to compute the histograms. The lines present in the plane for some of the maps (sine, cusp, cubic, and Ricker's population) correspond to those parameter values where the maps are periodic, but they quickly converge to the region of stochastic dynamics ($PE \approx 1$, $FIM \approx 0$).

We also explored the linear congruential generator, LCG, under the OP-FIM-PE plane. This map is known for being highly chaotic and difficult to distinguish from randomness, therefore it is sometimes used as a pseudo-random generator. Recently, Olivares et al.

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Fig. 6. Projection of the logistic map on the OP-FIM-PE plane for dimension D = 4. Four different sorting protocols are presented. (a) Lehmer. (b) Keller. For (c) we generated Gaussian noise and found a sorting protocol that minimized its FIM. (d) corresponds to the sorting protocol that minimizes FIM for the logistic map.

[30], via transforming the time series into a network using ordinal patterns, could discriminate the linear congruential generator from white noise.

On our OP-FIM-PE plane this map presents a very clear signature for the range $0 \le A \le 3$. While an uncorrelated stochastic process, such as white noise, would remain in the region where $PE \approx 1$ and $FIM \approx 0$, the linear congruential generator, for this A interval, covers a wide range of values $(0.1 \le PE \le 1; 0 \le FIM \le 0.95)$, and moves in a well definite narrow path (see Fig. 5). It overlaps part of the path covered by the logistic and other non-invertible maps, but covers a broader path on the plane, with clear signatures of structure in the complex dynamics. Unfortunately, for higher A values, in particular for A = 7141 as used in Ravetti et al. [9], or for the whole range $4 \le A \le 7200$, the projection of the LCG does not show distinctive signatures. It collapses in the plane to high entropy and low FIM. Nevertheless there are some parameter values for which the system moves up and to the left. This corresponds for numbers that are far from being prime numbers.

Another distinctive map is the Pinchers map $(x_{n+1} = tanh [S(x_n - C)]]$, with C = 0.5 and $0 \le S \le 4$). This map, even though it is also non-invertible, it does not present the same characteristic signature on the OP-FIM-PE plane as the previously studied maps. It unfolds a seven-shaped path that does not overlap with that of the logistic map, except in a small region of the plane (see Fig. 5(e)). Despite not sharing the same signature, the kink it presents at $S \approx 3$ on the plane unveils a change in the dynamics that is not revealed in the bifurcation map. By exploring the time series before and after the kink, we can see that there is an actual change in behavior (see Fig. 5(c,f)). The time series for control parameters S < 3 show a more unstructured nature, and are noticeable different from those for S > 3, which present more clear features.

In this study we have used Lehmer sorting protocol, as it is well known and used, and it is easy to generalize to higher dimensions. But we mentioned above that the way we order the ordinal patterns has an influence on the value of the computed FIM. For dimension D = 3 there are six ordinal patterns and 720 different sorting protocols. This can influence the way each map is por-

trayed on the OP-FIM-PE plane. We have computed other sorting protocols for the logistic map and found that some orders capture almost no details of the complex structure, while other orders do capture it, although with a different signature on the plane each (see Fig. 6). This raises the following questions: Is there an op-timum sorting protocol to unveil the complex structure for each chaotic map? Is there a sorting protocol for a lower dimension that extracts the same detail of structure than other sorting protocols at higher dimensions?

4. Conclusions

We have studied the Fisher–Shannon plane using ordinal patterns to characterize the complexity of various non-invertible maps. We have found that, transforming the time series of the dynamics into sequences of ordinal patterns, previous to locating the dynamical system on the plane, reveals their complex structure, and distinguishes the different regions for the different values of the control parameter. This differentiation is not present in the Fisher–Shannon plane when not computed with ordinal patterns, but with the probability distribution function of the time series. In this latter case the maps lie on the region of the plane non distinguishable from stochasticity.

Because the ordinal patterns extract temporal correlations in the time series, our results display regions with different temporal correlations or memory in the systems. The ordinal patterns can be calculated with different dimensions, which explores different temporal scales of the system, but for higher dimensions, it is required very long time series and high computational resources, as the number of possible patterns increases quickly. By studying the effect of the dimension of the ordinal patterns we have seen that OPs of low dimension do not extract all the complexity of the underlying dynamics, and higher dimensions are required. But dimensions $D \ge 6$ present the same qualitative results, and there is no need to consider higher dimensions, with what this implies in computational resources.

One important result found is that the structure projected in the OP-FIM-PE plane produces the same path for most of the non-

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invertible maps studied. This reveals similarities of the different maps that can also be tracked back to the bifurcation diagrams of the maps. This has not to be considered as a universal character as Pinchers map, and linear congruential generator do not show this structure; and cusp map lies in the same region but lacks the more complex structure of the rest of the maps. But this is a signature of common behavior of the complex dynamics of those non-invertible maps.

Although this technique is not able to distinguish the linear congruential generator for the highly chaotic regime (A > 3) it permits to distinguish the linear congruential generator from white noise for the small range $0 \le A \le 3$, for which it depicts a non-trivial structure in the plane even for values where it presents chaos.

Finally, we have shown how this technique allows to detect changes in the dynamics that other techniques do not. In the Pinchers map, the shape of this map on the OP-FIM-PE plane notices an angle at $S \approx 3$, that has no correlation in the bifurcation diagram. This change of trend is related to a change in the behavior of the time series.

It is known that the Fisher Information Measure presents sensitivity to the sorting of the ordinal patterns. We have used the Lehmer sorting protocol, but these maps present different shapes on the plane depending on the sorting criteria. A deep analysis of this particular is under way and will be presented elsewhere. Also of interest is expanding this study to other families of maps, such as conservative or dissipative maps.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

David Spichak: Software, Investigation, Formal analysis, Writing - review & editing. **Audrey Kupetsky:** Software, Investigation. **Andrés Aragoneses:** Conceptualization, Investigation, Supervision, Formal analysis, Validation, Funding acquisition, Writing - original draft, Writing - review & editing.

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