A 3D-Printed Wheel with Constant Mass and Variable Moment of Inertia for Lab and Demonstration

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We present a versatile experimental apparatus for exploring rotational motion through the interplay between the moment of inertia, torque, and rotational kinetic energy of a wheel. The heart of this experiment uses a 3D-printed wheel along with easily accessible stock components that allow for the adjustment of the moment of inertia while keeping the total mass of the wheel constant. The wheel can act as a massive pulley of variable moment of inertia that allows students to measure the moment of inertia of the bare wheel by applying a constant torque to the system. The wheel can also be used to explore rotational kinetic energy in the form of races down ramps. The 3D-printed aspect of this wheel allows anyone with access to a 3D printer to create, explore, and modify this wheel at a low cost, allowing for more flexibility and accessibility for student and instructor exploration and modification. In the study of linear kinematics, it is easy to investigate how systems evolve with variable mass. With rotational motion the situation is more complicated due to the fact that the moment of inertia depends not only on the mass, but also on the distribution of the mass. This can be demonstrated in the form of massive pulleys in Atwood machines[1] and in a rolling race of objects of varying mass and moment of inertia. For lab explorations it is best to isolate a single variable that can be changed. For rotational motion it is difficult to find a wheel with constant mass but different moments of inertia. There do exist examples that solve this problem,[2,3] but they require workshop access and cannot act as pulleys. To address this issue we have designed a wheel whose moment of inertia can easily be created and manipulated while the mass of the system remains constant, providing an accessible, flexible, and robust apparatus to explore the interplay between moment of inertia and torque and rotational kinetic energy.

The apparatus we designed is a 3D-printed wheel with various locations where steel dowel pins can be inserted to adjust the moment of inertia of the system while keeping the mass fixed. 3D printers are becoming more accessible in high schools, colleges, and universities, and public libraries. This accessibility eliminates the need to have access to a full machine shop for production. The 3D-printed design is open source and will hopefully evolve as students and instructors provide feedback. Assuming access to posts and post clamps, the experimental apparatus is affordable and accessible to classrooms everywhere.

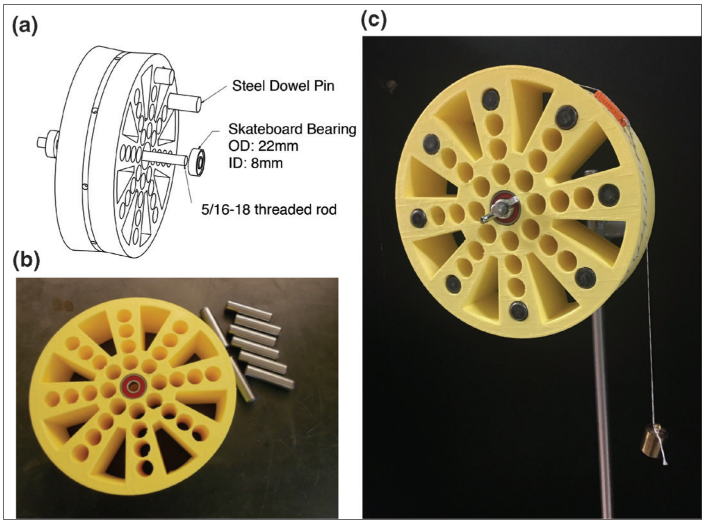

Fig. 1. Wheel experiment for investigating moment of inertia. (a) Schematic drawing of the assembly of the wheel. The dimensions shown are for parts that are consistent among different variations of the wheel. Not shown are the retaining nuts and washers as well as the stand for mounting the rod. (b) Picture of the printed wheel with a thickness of 2 in, outside diameter of 6.5 in, and the eight 0.5-in dowel pins. (c) The full experimental setup showing the mounted wheel and the dowel pins inserted at the largest radial location. A 3D-printed jig (orange in the figure) is used to secure the line in the mounting holes in the rim of the wheel.

The apparatus

The heart of the device is a 3D-printed wheel with eight spokes where steel dowel pins can be inserted at various radii, as shown in Fig. 1. For this investigation we printed our wheel with a Lulzbot TAZ 6 printer with PLA material at a 20% fill density. For our investigation we printed a wheel with a 6.5-in outside radius and used 0.5-in radius steel dowel pins that were 2 and 2.5 in long. The length of the dowel pin does not need match the thickness of the wheel as long as the retaining nut used on the threaded rod does not impede rotation. The wheel can be scaled to various sizes by modifying the design files. At all scales there are eight spokes evenly distributed angularly about the central axis. There are four radial positions in our configuration. The radial edge of the wheel has a small taper that gives two contact points for rolling-cylinder investigations. The radial edge has some holes in it that allow for a string to be attached via a paper clip or 3D-printed jig. At the center of each side there is a place for a 22-mm bearing (standard skateboard bearing), which comprises the hub of the unit. Using a piece of 5/16-in threaded rod for the axle, the pulley is complete as shown in Fig. 1. All of the parts of this apparatus, along with detailed instructions on assembly, can also be found on the Thingiverse online repository.[4]

The part list and approximate unit cost for one complete setup is:

- One 3D-printed wheel (ABS or PLA). One wheel with a diameter of 6.5 in and a thickness of 2 in.: ~ $15
- Eight dowel pins: 0.5-in diameter, 2-in length (for shown setup): ~ 8 × $1.8 = $14.40
- Two outside diameter 22 mm, inside diameter 8 mm bearings (standard skateboard bearing): ~ $1.05 for two bearings
- One 5/16-in or 8-mm threaded rod for mounting: ~ $1.50
- Two mounting hardware: ~ $1

Using common post mounts and clamps to secure the threaded rod, the per unit price of the apparatus becomes less that $35. This could outfit an eight-team lab for less than $200.[5] We also chose to buy the dowel pins in order to eliminate the need for a machine shop. If standard grade and on-hand materials are used, then the cost can be pushed even lower to less than $25 a setup, depending on the setup.

The experiment

In our lab we first asked the students to calculate the rate of gravitational acceleration g and moment of inertia I_{wheel} of the wheel by using the setup shown in Fig. 2. This is done by using the wheel as a pulley and varying the moment of inertia by redistribution of the masses. For each configuration a small mass m_0 is suspended from thread or fishing line that is wound around the pulley (see Fig. 2). By measuring the time and distance the mass falls, an accurate measurement of the acceleration a can be found. When the mass is released, it accelerates downward at a given by Newton's second law,

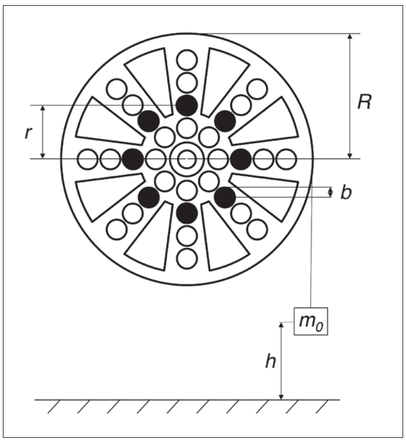

Fig. 2. Diagram of how the wheel of radius *R* is used in our experiment to measure *g* and the moment of inertia of the wheel I_{wheel}. Dowel pins (black) of radius *b* are inserted into the wheel at a radial position *r*. A small mass m_0 on a string is wound up on the wheel and is suspended a height *h* above the floor or table. Once the mass is released, the wheel will rotate and the mass will accelerate downward at *a*. By measuring how long it takes for the mass to fall for various radial positions of dowel pins, *g* and I_{wheel} can be measured.

$$m_{0g} - T = m_0 a, \tag{1}$$

where g is the acceleration due to gravity, a is the acceleration of the mass, m_0 is the suspended mass, and T is the tension in the string. The acceleration can be found by measuring the time it takes for the released mass to fall a measured distance. The tension force also exerts a torque τ on the wheel such that

$$T \cdot R = I_{tot}\ \alpha, \tag{2}$$

where R is the radius of the wheel, I_{tot} is the total moment of inertia of the apparatus, and α is the angular acceleration of the wheel. While the string remains taut, the constraint that $a = \alpha R$ holds. Using the parallel-axis theorem, I_{tot} can be written in terms of the moment of inertia of the wheel, I_{wheel}, the moment of inertia of each dowel pin, $I_{dp} = 1/2\ m_{dp} b^2$, and the radial location of each dowel pins r;

$$I_{tot} = I_{wheel} + \sum n_i \left[\tfrac{1}{2} m_{dp} b^2 + m_{dp} r^2\right], \tag{3}$$

where m_{dp} is the mass of each dowel pin, b is the radius of the dowel pins, and the sum is done for each radial position, and n_i is the number of dowel pins located at that radial position. In our experiment we placed n dowel pins at the same radius, simplifying the above to be

$$I_{tot} = I_{wheel} + n\left[\tfrac{1}{2} m_{dp} b^2 + m_{dp} r^2\right]. \tag{4}$$

With Eqs. (1), (2), and (4) along with kinematics to relate the a, the distance the mass falls h, and time it took to fall t, we arrive at a relationship between the radial position of the dowel pins and time in the form of

$$\left[\frac{m_0 g R^2}{2hnm_{dp}}\right] t^2 = r^2 + \frac{m_0 R^2}{nm_{dp}} + \frac{I_{wheel}}{nm_{dp}} + \frac{b^2}{2}. \tag{5}$$

Equation (5) can be linearized by plotting t^2 vs. r^2, which yields the first goal of this lab. The students are instructed to vary r and measure the time t it takes for the mass to fall a distance h. From the linearization of Eq. (5) and scaling the t^2 by the known coefficients, the slope of the plot directly yields $1/g$, which allows for a measurement of local acceleration due to gravity. The intercept of the graph can be used to solve for the moment of inertia of the wheel. The results are shown in Fig. 3. The average of the results obtained by the students are reasonable within experimental errors: $g = (10.16 \pm 0.34)$ m/s^2 for the eight dowel pins investigation and $g = (8.99 \pm 0.41)$m/s^2 for the four dowel pins investigation. In our investigations we find that $g = (9.82 \pm 0.15)$ m/s^2 for the eight dowel pins investigation and $g = (10.15 \pm 0.19)$ m/s^2 for the four dowel pins investigation. The moment of inertia is calculated from the intercept of the graph; the value the students obtained was $I_{wheel} = (1.01 \pm 0.08) \times 10^{-3}$ kg·m^2 and our value was $I_{wheel} = (1.13 \pm 0.04) \times 10^{-3}$ kg·m^2. The theoretical value for the moment of inertia of a solid and hollow cylinder of the same dimensions and mass to the 3D-printed one are 0.99×10^{-3} kg·m^2 and 1.98×10^{-3} kg·m^2, respectively. The measured value for our wheel is between both of these, as expected.

A determination of the moment could be done via computer modeling and would be an interesting extension of the lab, but the cross-hatch in-fill of the printing process makes

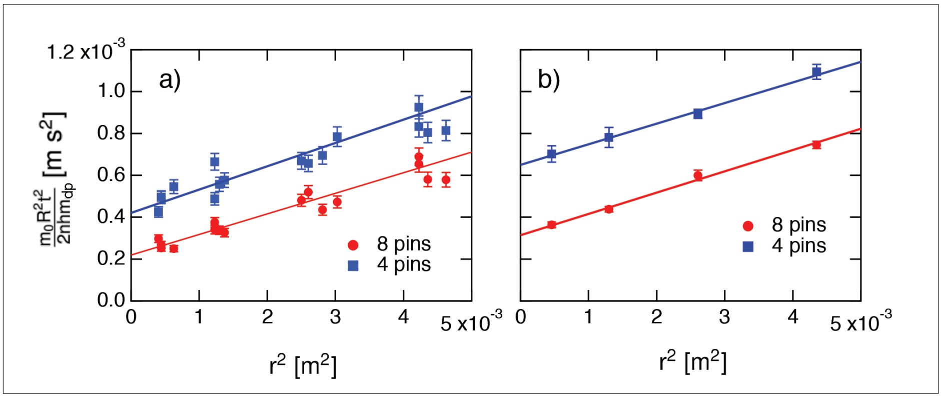

Fig. 3. A linearized plot of our experimental data of (a) five student groups and (b) our trial runs. The student data used a drop mass of 10 g and we used a drop mass of 20 g. The student's drop height was in the range of 1.5 m to 1.8 m and our drop height was 1.4 m. The horizontal axis is scaled such that the slope yields a measurement of 1/*g*. The dowel pins were evenly spaced for the four-pin investigation.

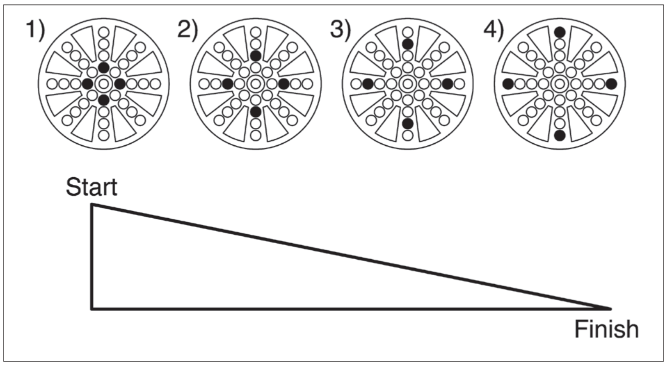

Fig. 4. Example of how our wheel can be utilized to perform the rotational kinetic energy race. The four wheels all have equal mass yet different moments of inertia. If they all are released at the start of the ramp at the same time, which distribution will reach the finish first?

it hard to model the exact density distribution as well as the exact moment of inertia of the wheel. This method demonstrates how to use experimental techniques to determine physical quantities that are difficult to computationally and analytically determine.

This result provides the template for measuring the moment of inertia of arbitrary objects. Now that the moment of inertia of the wheel is known, an object of unknown moment of inertia can be attached to the wheel and the experiment performed again. The wheel also provides a demonstration of the rotational kinetic energy race.[6] In this race objects of equal masses but different moments of inertia roll down a ramp and the student is tasked to determine which object will finish first. One configuration of how our wheel can be used in this manner is shown in Fig. 4. To arrive at the correct answer, the student must understand the relationship between potential energy and translational and rotational kinetic energy. Intuition on this subject is notoriously shaky,[7] but this setup is a clear example for students to explore and understand the impact that moment of inertia plays. We have also used this wheel for demonstrations in lecture to show both the consequences of a massive pulley and as the kinetic energy race in Fig. 4. While we outline one experiment in this report, this wheel could be used in a variety of different ways. With the variety of configurations of moment of inertia possible, other challenges to the students can be raised such as, "Can you have three wheels with different masses reach the bottom of the ramp at the same time?" This could be answered analytically, computationally, and experimentally with this setup.

This experiment shows flexibility and utility of 3D-printed apparatuses to explore concepts that are difficult for students to explore with traditional manufacturing processes. At the same time the 3D-printed aspect of it allows the wheel to easily be modified by the students or instructor and have those modifications become reality without the need of a dedicated machinist or machine shop. This apparatus has filled a void in our lecture/demo and lab toolbox, and gives the students experience with rotational motion by only varying the moment of inertia while keeping a constant mass.

References

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