General Relativity and Gravitation, Vol. 30, No. 2, 1998

The Growth of Cosmological Perturbations in the Transition Eras

Andrés Aragoneses,[1] Diego Pavón[1] and Winfried Zimdahl[2]

Received April 14, 1997

In this article we determine the dominating modes of the cosmological perturbations in different transition eras of the universe evolution (namely, radiation-dust, dust-vacuum, dust-K-matter, and K-matter-vacuum) in a covariant and gauge-invariant manner.

KEY WORDS : Evolution of inhomogeneities

1. INTRODUCTION

According to the conventional picture the formation of cosmic structures can be traced to the existence of small initial inhomogeneities in the early Universe which, whatever their origin was, grew with time. As the Universe expanded these inhomogeneities necessarily felt this effect, as it tends to dilute them and consequently to homogenize the Universe. So, any inhomogeneity that we may observe today has necessarily survived the cosmic expansion thanks to the natural tendency of matter (and radiation) to clump by gravitational attraction. Therefore the perturbations will either grow or decay depending on the type of matter considered in conjunction with the specific expansion law. Since the pioneering work of Lifshitz [1] much attention has been paid to the evolution of small inhomogeneities in

[1] Departamento de Física, Facultad de Ciencias, Universidad Autónoma de Barcelona, 08193 Bellaterra, (Barcelona) Spain.
E-mail: andres@ulises.uab.es , diego@ulises.uab.es
[2] Fakultät für Physik, Universität Konstanz, PF 5560 M678 D-78457 Konstanz, Germany. E-mail: zimdahl@spock.physik.uni-konstanz.de

0001-7701/ 98/ 0200-0299$15.00/ 0

the radiation-dominated era as well as in the matter-dominated one (see for instance Refs 2–4). By contrast, the transition era from radiation to dust, in which none of these components dominated the expansion, has received comparatively little attention (see, however, Ref. 5). The target of this paper is to study the evolution of small inhomogeneities in an almost Friedmann–Lemaître–Robertson–Walker (FLRW) universe with flat space sections in different transition eras, firstly, in the one just mentioned, secondly, in the transition era from dust to a vacuum-dominated universe (i.e. a universe in which a small cosmological constant, $\Lambda > 0$, is assumed to exist), thirdly in the transition era from dust to a K-matter dominated expansion (K-matter came into fashion some years ago; Ref. 6), and finally, from K-matter to a vacuum-dominated universe. To carry out this analysis we shall resort to the covariant and gauge-invariant cosmological perturbation theory put forward by Olson [7], Woszczyna and Kulak [8] and Ellis and coworkers [9]. We will rely especially on the pedagogical and elegant presentation of Jackson [10]. Some of these eras may never show up. It may happen that the cosmological constant is exactly zero, and/or that the K-matter does not exist in reality. At any rate, it is always interesting to anticipate the evolution of the small inhomogeneities in the event that this were not the case.

2. PERTURBATION THEORY

It is advantageous to study the evolution of small inhomogeneities in the Universe in terms of covariant and gauge-invariant quantities. While the gauge-invariant approach by Bardeen [11] aims to overcome the ambiguity related to the splitting of the spacetime metric and stress-energy tensor quantities into a zeroth-order and small first-order perturbations, the covariant approach, especially worked out by Ellis *et al.* [9], does not introduce a fictitious background universe at all. In this spirit we shall consider the covariantly defined spatial gradient of the energy density ρ, i.e. $h^{\upsilon}_{\mu}\rho_{,\upsilon}$, where $h^{\upsilon}_{\mu} \equiv g^{\mu}_{\upsilon} + u^{\mu}u_{\upsilon}$, with $u^{\mu}u_{\mu} = -1$, is the spatial projector on the comoving hypersurfaces. More precisely we will focus our attention on the fractional density gradient on comoving hypersurfaces used by Jackson,

$$\mathcal{D}_{\mu} \equiv \frac{a h^{\upsilon}_{\mu}\rho_{,\upsilon}}{\rho + P} \qquad (\mu, \upsilon = 0, 1, 2, 3), \tag{1}$$

where P is the pressure, a is a length scale, generally given by $u^{\mu}_{;\mu} \equiv \Theta = 3\dot{a}/a$. For an almost homogeneous and isotropic universe a coincides with the scale factor of the Robertson–Walker metric. In the latter case the second-order differential equation that governs the evolution of $\mathcal{D}^\mu$ [see Ref. 10, eq. (57)] is

$$\ddot{\mathcal{D}}^\mu + \left(\frac{2}{3} - \frac{dP}{d\rho}\right)\Theta\dot{\mathcal{D}}^\mu - \left[\left(\frac{dP}{d\rho}\right)^{\cdot}\Theta + [4\pi G(\rho - 3P) + 2\Lambda]\frac{dP}{d\rho} + 4\pi G(\rho + P) + \frac{dP}{d\rho}\frac{\nabla^2}{a^2}\right]\mathcal{D}^\mu = 0. \qquad (2)$$

Assuming the factorization $\mathcal{D}_\mu = \mathcal{D}_{(m)}(t)\nabla_\mu Q_{(m)}$, where $Q_{(m)}$ satisfies

$$\nabla^2 Q_{(m)} = -m^2 Q_{(m)},$$

one finds in the spatially flat FLRW case, where m is related to the physical wavelength λ by $\lambda = 2\pi a/m$, that the last term in eq. (2) can be neglected on large perturbation scales, i.e. for $m \ll 1$ (see Ref. 12 and references therein). In such a case the solutions for radiation and dust can be found analytically. These are $\mathcal{D}_{(m)(\mathrm{radiation})} \propto a^2$ and $\mathcal{D}_{(m)(\mathrm{dust})} \propto a$, for the growing modes, and $\mathcal{D}_{(m)(\mathrm{radiation})} \propto a^{-1}$ and $\mathcal{D}_{(m)(\mathrm{dust})} \propto a^{-3/2}$ for the decaying ones. But for a mixture of both components — see below — no analytical solution exists. Since we will only be interested in large perturbation scales throughout this paper we will omit any index attached to $\mathcal{D}$ in the following.

3. THE RADIATION PLUS DUST ERA

To describe the evolution of the Universe from very early times to our days we consider an almost FLRW universe of flat space sections filled with mixture of massless radiation and dust such that there is no net interchange of energy between both components. The mixture begins expanding like radiation and then gently changes its expansion law to that of dust. To implement this we take an equation of state that depends on the scale factor (see Ref. 13)

$$P(a) = [\gamma(a) - 1]\rho(a) \qquad (3)$$

where

$$\gamma(a) = \frac{a + (4/3)a_*}{a + a_*}, \qquad (4)$$

a_* being the scale factor corresponding to the instant $\rho_{\mathrm{dust}} = \rho_{\mathrm{radiation}}$. By solving the zeroth-order Friedmann equation with $k = 0$, i.e.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\kappa\rho,$$

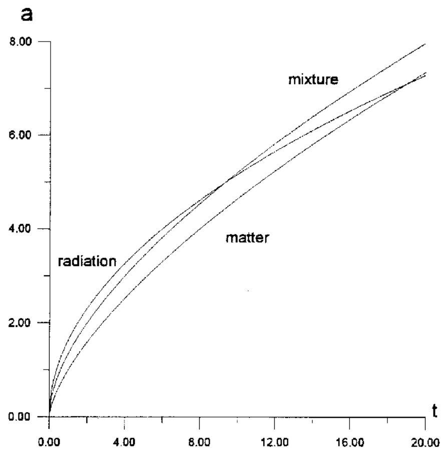

Figure 1. Evolution of the scale factor for FLRW universes dominated, respectively, by radiation, dust, and a mixture of radiation and dust with equation of state given by (3) and (4).

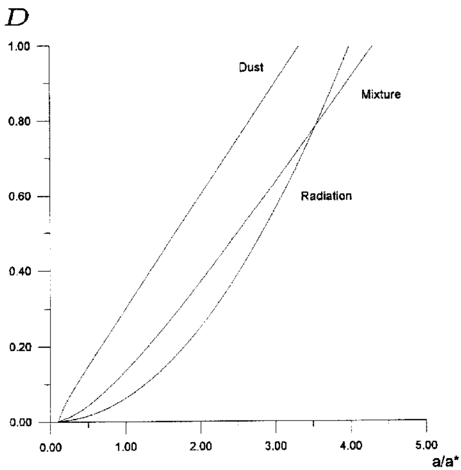

Figure 2. Growing modes of the cosmological perturbations for radiation, dust, and the mixture of radiation and dust.

where as usual $\kappa \equiv 8\pi G$, we obtain the relation between the time and the scale factor

$$t(a) = (a - 2a_*)\sqrt{a + a_*} + 2a_*\sqrt{a_*} \, . \tag{5}$$

Figure 1 depicts the evolution of the scale factor for radiation, dust, and the mixture radiation-dust. For the latter fluid eq. (2) takes the form

$$\ddot{\mathcal{D}} + A\frac{2a + a_*}{a^2\sqrt{a + a_*}}\dot{\mathcal{D}} - \frac{A^2}{2a^4(a + a_*)}(3a^2 + 6aa_* + 4a_*^2)\mathcal{D} = 0,$$

where $A \equiv (a_0^2 H_0)/(\sqrt{a_0 + a_*})$ with $H_0 \equiv (\dot{a}/a)_0$ (here the subindex zero means present time). Solutions to this equation can be found only numerically. There are two modes for the evolution of the perturbations, a growing mode and a decaying one. Only the former is of interest to us since the latter cannot lead to structure formation. Graphs for the evolution of $\mathcal{D}$ in terms of the scale factor for radiation, matter and the mixture are shown in Figure 2. For $a(t) \geq a_*$ the slopes of the curves for dust and the mixture coincide, as it should.

4. THE DUST PLUS QUANTUM VACUUM ERA

Above we have considered a universe with a vanishing cosmological constant [$\Lambda = 0$ in eq. (2)], something very reasonable because, if it is non-vanishing, it must be so small that it has had no noticeable impact on cosmic evolution so far. However it is perfectly admissible to consider a positive non-zero cosmological constant as it amounts to a non-zero vacuum energy density. This may well be the remnant left over after a non-complete vacuum decay which supposedly took place during the inflationary period (there are some papers dealing with this possibility; Ref. 14). In this case, as ρ_v = constant and $\rho_{\rm dust} \propto a^{-3}$, there will be an instant such that $\rho_v \simeq \rho_{\rm dust}$, and from this time onward the vacuum energy will dominate. Here we consider the transition from a dust-dominated to Λ-dominated universe as well as the corresponding cosmological perturbations at that epoch.

For a dust-vacuum fluid the energy conservation equation can be written as

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + P) = -3\frac{\dot{a}}{a}a\gamma\rho \tag{6}$$

with

$$\rho = \rho_m + \rho_v\,,$$

$$P = P_v = -\rho_v\,,$$

$$\gamma(a) = \frac{n a_0^3}{n a_0^3 + a^3}\,,$$

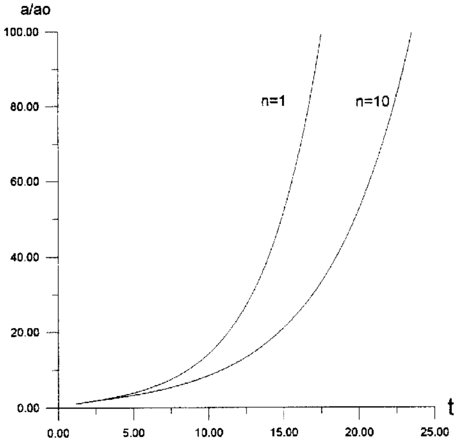

Figure 3. Evolution of the scale factor of a FLRW universe with dust plus vacuum energy.

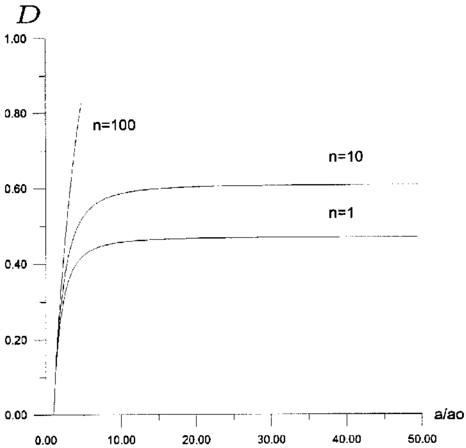

Figure 4. Growing mode of the cosmological perturbations for a FLRW universe with dust plus vacuum energy.

$$
\rho = \rho_0 \frac{a_0^3}{a^3}\left(\frac{n a_0^3 + a^3}{(1+n)a_0^3} \right),
$$
$$
n = \frac{\rho_{m\,0}}{\rho_v} . \tag{7}
$$

Note that the last expression implies the right asymptotic limits, i.e. $\rho \propto a^{-3}$ and $\rho \propto$ constant for small and for large scale factors, respectively.

For an almost FLRW universe with $k = 0$ the solution to the zeroth-order Friedmann equation is

$$t = \frac{2}{3\sqrt{A}} \ln \left[\frac{\sqrt{Aa^3 + B} + \sqrt{Aa^3}}{\sqrt{B}} \right], \tag{8}$$

where

$$A = \frac{B}{na_0^3}, \qquad B = \frac{\kappa n \rho_0 a_0^3}{3(1+n)}. \tag{9}$$

Figure 3 depicts the scale factor versus the time for different values of Λ. As expected, the higher the contribution of dust, the slower the expansion.

On large perturbation scales eq. (2) reduces to

$$\ddot{\mathcal{D}} + 2\sqrt{\frac{\Lambda}{3}\frac{a^3 + na_0^3}{a^3}}\,\dot{\mathcal{D}} - \frac{n\Lambda}{2}\left(\frac{a_0}{a}\right)^3 \mathcal{D} = 0, \tag{10}$$

where we have used the relationships

$$\Lambda = 8\pi G\rho_v, \qquad B = \frac{n\Lambda}{3}a_0^3. \tag{11}$$

We have numerically solved (10) for different n values. Figure 4 displays the corresponding graphs for dust perturbations. These show a steady initial growth, $\mathcal{D} \sim a$, followed by a gentle approach to a constant asymptotic value. The larger n, the higher the asymptotic value. This was to be expected on physical grounds. There are two competing effects, on the one hand the gravitational attraction of the matter on itself which for a static universe leads to an exponential growth, and on the other hand the Universe expansion, which for a vacuum-dominated universe turns out to be exponential. Therefore after a period, whose length depends on the ratio n between the energy densities, one effect offsets the other. Obviously, the asymptotic behavior for large $a(t)$ can also be found analytically from eq. (2) under the assumption $\Lambda \gg 8\pi G\rho_m$.

5. THE DUST PLUS K-MATTER ERA

The existence of K-matter, a form of energy that redshifts with cosmic expansion as a^{-2} and obeys an equation of state $P_K = -\rho_K/3$ (such as cosmic strings and some kinds of textures) has been postulated in the literature, and various of its cosmological consequences explored (see for instance Refs. 6 and 15). It may correspond to a topologically stabilized scalar-field configuration such as a net of not intersecting cosmic strings [16] or textures. Here we focus our attention in a homogeneous and isotropic universe in a phase of expansion such that its energy density is roughly evenly distributed between dust and K-matter (either type 1 or 2). We assume that the energy density of the massless radiation fluid and that of the vacuum can be ignored at this stage. Therefore $\rho = \rho_m + \rho_K = (B/a^3) + (A/a^2)$ and $P_K = -A/(3a^2)$, where A and B are positive definite constants. Applying the same ansatz as before we get $P(a) = [\gamma(a) - 1]\rho(a)$ with

$$\gamma(a) = \frac{2Aa + 3B}{3(Aa + B)} .$$

When this equation is combined with the energy conservation equation

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + P),$$

there follows

$$\frac{\rho}{\rho_0} = \left(\frac{Aa + B}{Aa_0 + B}\right)\left(\frac{a_0}{a}\right)^3 .$$

(Here the subindex zero refers to a suitably chosen time within this era.) This expression has the right asymptotic limits, i.e. $\rho \propto a^{-3}$ and $\rho \propto a^{-2}$ for the dust-dominated and K-matter dominated eras, respectively.

Likewise inserting $\gamma(a)$ in the zeroth-order Friedmann equation above for the flat case we obtain after integration

$$t(a) = \sqrt{\frac{3}{8\pi G}}\left[\frac{\sqrt{(Aa + B)a}}{A} - \frac{B}{\sqrt{A^3}} \ln \frac{(\sqrt{Aa + B} + \sqrt{Aa})}{\sqrt{B}}\right]. \qquad (12)$$

Again this equation has the right asymptotic limits, i.e. $t \propto a^{3/2}$ and $t \propto a$ for the dust-dominated and the K-matter dominated eras, respectively. From it we have for the deceleration parameter $q = -1 + \frac{3}{2}\gamma(a)$. In the case at hand eq. (2) reduces to

$$\ddot{\mathcal{D}} + \sqrt{\frac{8\pi G}{3a^3}}\left(\frac{3Aa + 2B}{\sqrt{Aa + B}}\right)\dot{\mathcal{D}} - 8\pi G\frac{B(2Aa + 3B)}{6(Aa + B)a^3}\mathcal{D} = 0 . \qquad (13)$$

We have numerically integrated this equation for different values of the ratio $n = (\rho_m/\rho_K)_0$. The corresponding behavior (Figure 5) is similiar to the one encountered in the dust plus vacuum situation studied earlier. After an initial growth the perturbations get frozen for large $a(t)$. The

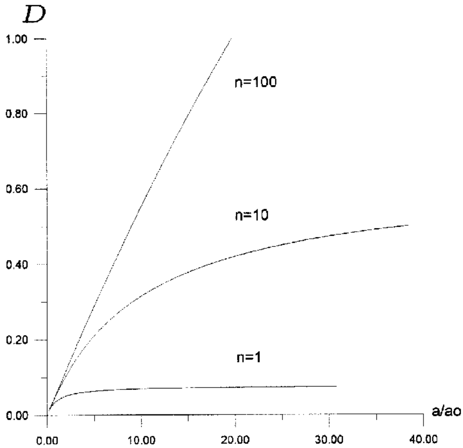

Figure 5. Growing mode of the cosmological perturbations for a FLRW universe with a mixture of dust plus K-matter. The higher n, the higher the constant asymptotic value of the fractional density gradient.

asymptotic behavior, $\mathcal{D} =$ constant, follows directly from the above equation by setting B to zero, and coincides with the analytical result from (2), with $\Lambda = 0$, when only K-matter is considered. It is worthy of note that our result for the expansion era dominated by the K-matter clearly differs from Kolb's (see eq. (45) in Ref. 6). This author finds a growth of the fractional density perturbations $\propto t^{(\sqrt{6}-1)/2}$ in this era. The difference with our result, i.e. $\mathcal{D} =$ constant, arises because this author uses a Newtonian approach, specifically eq. (15.9.23) of [2]. If one uses instead the corresponding relativistic generalization, namely eq. (15.10.57) of [2] and takes into account $dP_K/d\rho_K = -\frac{1}{3}$, one obtains a constant fractional density perturbation in line with our finding.

6. THE K-MATTER PLUS QUANTUM VACUUM ERA

Once a sufficiently long time has elapsed the influence of the dust component on the cosmic evolution may be ignored. If a cosmological constant really exists, after the dominance of the K-matter component the Λ term will take over. Here we consider the transition from a K-matter dominated expansion to a vacuum-dominated one. Using the corresponding expressions for both forms of energy (K-matter and vacuum) in the Friedmann equation and integrating the resulting expression one arrives

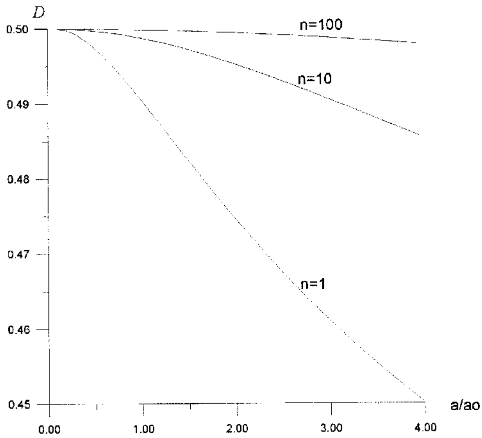

Figure 6. Dominating mode of the cosmological perturbations for a FLRW universe with K-matter plus vacuum energy.

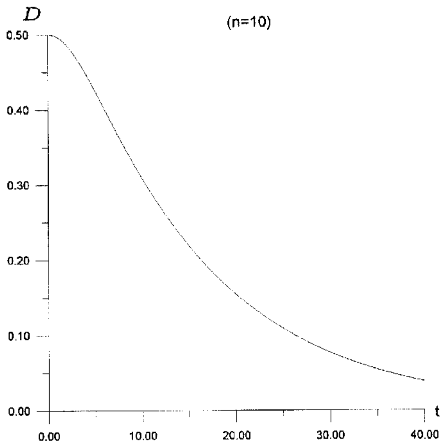

Figure 7. Same as in Figure 6, but now D is depicted in terms of time. Here it is more clearly seen that the cosmological perturbations vanish altogether after a sufficiently long time has elapsed.

at

$$t(a) = \sqrt{\frac{3}{\Lambda}} \ln\left[\sqrt{\frac{\Lambda}{\kappa A}}\, a + \sqrt{1 + \frac{\Lambda}{\kappa A} a^2}\right]. \tag{14}$$

Notice that in this case $\ddot{a} \propto a$. Moreover $q = -(\Lambda a^2)/(8\pi G A + \Lambda a^2)$, and therefore for $\Lambda a^2 \gg 8\pi G A$ one has $q \rightarrow -1$ and $q \rightarrow 0$ in the opposite limit, as it should. In the case at hand the differential equation (2) governing the perturbations reduces to

$$\ddot{\mathcal{D}} + \sqrt{3\Lambda\left(\frac{a^2 + n a_0^2}{a^2}\right)}\,\dot{\mathcal{D}} + \frac{\Lambda}{3}\mathcal{D} = 0, \tag{15}$$

where $n \equiv \rho_{K\,0}/\rho_v$. As Figures 6 and 7 show, for small $a(t)$ the perturbations in the K-matter fluid stay constant and then decay to die away for large $a(t)$. Both asymptotic limits can be directly guessed by direct inspection of (15).

7. CONCLUDING REMARKS

We have explored the evolution of the dominating modes of the cosmological perturbations during some transition eras of the Universe expansion. To do this we first determined in each case the dependence of the scale factor on t and introduced $a(t)$ into Jackson's equation (2), along with the state equations of the fluids relevant to the transition era under consideration. Equations (12)–(15) are new as well as Figures 2 and 4–7. Figures 1 and 3 have been included for the sake of completeness.

As expected in the transition era from radiation-dominated to dust-dominated expansion the growing mode increases faster than in the pure radiation case and slower than in the pure dust one. In the transition era dominated by dust plus quantum vacuum the mode grows until the fluid reaches the exponential expansion regime. From that point onwards $\mathcal{D}$ stays constant at a value that depends on the ratio $\rho_{m\,0}/\rho_v$. Likewise at the end of the transition era dust plus K-matter the growing mode freezes but now the scale factor varies asymptotically as t only. Earlier results predicting a growth of cosmological perturbations in a K-matter universe were corrected. Finally, during the transition era from K-matter to quantum vacuum-dominated expansion, initially the mode remains constant, then decays once the expansion becomes dominated by the vacuum energy, and eventually dies away. So strictly speaking there is not a growing mode.

ACKNOWLEDGEMENTS

This work has been partially supported by the Spanish Ministry of Education (grant PB94-0718), the NATO (grant CRG940598) and the Deutsche Forschungsgemeinschaft. A conversation with Josep Triginer is also gratefully acknowledged.

REFERENCES

1. Lifshitz, E. M. (1946). *J. Phys. (Moscow)* 10, 116.
2. Weinberg, S. (1972). *Gravitation and Cosmology* (Wiley, New York).
3. Padmanabhan, T. (1995). *Formation of Structures in the Universe* (Cambridge, Cambridge University Press).
4. Coles, P., and Lucchin, F. (1995). *Cosmology. The Origin and Evolution of Cosmic Structure* (J. Wiley, New York).
5. Peebles, P. J. E., and Yu, J. T. (1970). *Astrophys. J.* 162, 815.
6. Kolb, E. W. (1989). *Astrophys. J.* 344, 543.
7. Olson, D. W. (1976). *Phys. Rev. D*14, 327.
8. Woszczyna, A., and Kulak, A. (1989). *Class. Quantum Grav.* 6, 1665.
9. Ellis, G. F. R., and Bruni, M. (1989). *Phys. Rev. D*40, 1804; Ellis, G. F. R., Hwang, J., and Bruni, M. (1989). *Phys. Rev. D*40, 1819; Ellis, G. F. R., Bruni, M., and Hwang, J. (1990). *Phys. Rev. D*42, 1035.
10. Jackson, J. C. (1993). *Mon. Not. R. Astron. Soc.* 264, 729.
11. Bardeen, J. M. (1980). *Phys. Rev. D*22, 1882.
12. Zimdahl, W. (1997). Preprint gr-qc/9707047, to appear in *Class. Quantum Grav.*
13. Méndez, V., and Pavón, D. (1996). *Mon. Not. R. Astron. Soc.* 282, 753.
14. Taylor, T. R., and Veneziano, G. (1989). "Quenching of the cosmological constant", Report CERN-TH5433/89; Frewin, R. A., and Lidsey, J. E. (1993). *Int. J. Mod. Phys. D*2, 323; Lima, J. A. S., and Maia, J. M. F. (1994). *Phys. Rev. D*49, 5597.
15. Kamionkowski, M., and Toumbas, N. (1996). *Phys. Rev. Lett.* 77, 587.
16. Gott, III, J. R., and Rees, J. M. (1987). *Mon. Not. R. Astron. Soc.* 227, 453.